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Asymptotic theory of turbulent shear flows

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A theory is proposed in this paper to describe the behaviour of a class of turbulent shear flows as the Reynolds number approaches infinity. A detailed analysis is given for simple representative members of this class, such as fully developed channel and pipe flows and two-dimensional turbulent boundary layers. The theory considers an underdetermined system of equations and depends critically on the idea that these flows consist of two rather different types of regions. The method of matched asymptotic expansions is employed together with asymptotic hypotheses describing the order of various terms in the equations of mean motion and turbulent kinetic energy. As these hypotheses are not closure hypotheses, they do not impose any functional relationship between quantities determined by the mean velocity field and those determined by the Reynolds stress field. The theory leads to asymptotic laws corresponding to the law of the wall, the logarithmic law, the velocity defect law, and the law of the wake.

1. Introduction

The importance of turbulent shear flows is well recognized in fluid flow problems of aeronautical, chemical, civil, and mechanical engineering. Expositions of the current state of knowledge can be found in the works of Phillips (1969), Kline *et al.* (1967) and Rotta (1962, pp. 3–219), and the earlier contributions of Townsend (1956), Clauser (1956) and Coles (1956) continue to be of great interest.

Much of what is known about turbulent shear flows stems from experimental data. Dimensional and similarity arguments have been employed to obtain general empirical correlations. The attempts to integrate this empirical information with the general equations of turbulent flows have not been fully satisfactory. The limitations of the attempts originate from the closure hypotheses employed to make the set of governing equations a fully determined system. All too often, the hypotheses do not describe any fundamental property of the mechanics of the flows. In some cases, such as the eddy viscosity hypotheses, their soundness and validity have been questioned on the basis of general physical considerations.

The theory proposed in subsequent sections considers the entire region of flow and deliberately avoids similarity, dimensional or eddy viscosity arguments. Instead, the method of matched asymptotic expansions, whose power has been well demonstrated in laminar flows, is applied to turbulent flow problems. A novel feature of the theory is that it deals with an underdetermined system of equations. The asymptotic hypotheses employed in the theory describe the orders of various terms in the equations of mean motion and of the kinetic energy of fluctuating motion. They are considerably milder than the closure hypotheses as they do not impose any functional relationship between quantities determined by the mean velocity field, such as mean rate of deformation, and quantities determined by the Reynolds stress field, such as any component of the stress or its spatial derivative.

The theory leads to asymptotic laws of the same form as the law of the wall, the velocity defect law and the logarithmic law. Since these empirical correlations depend on the parameters of the problem in a limited manner, they are sometimes referred to as universal laws. However, the empirical laws are necessarily approximate and small systematic departures are known in some cases. It is therefore preferable to regard them as asymptotic laws in the sense that they become exact in the limit as the Reynolds number tends to infinity. Incidently, this interpretation would account for the appearance of constants, such as von Kármán's constant, which do not depend on the parameters of the problem.

The crucial idea of the present theory is that there are two layers having rather different properties and there are two length scales describing their thickness. The thinner layer close to the wall is the one in which viscous stresses remain significant in the high Reynolds number limit. The production of turbulent kinetic energy from the mean flow and its diffusion due to the fluctuating motion remain equally significant or are of the same order in this layer. The outer layer in which Reynolds stresses alone remain significant is on the other hand very much thicker. This feature of turbulent shear flows has been well recognized and is known to be physically sound.

The treatment of turbulent boundary layers presented here is general enough to accommodate arbitrary initial conditions and arbitrary pressure distributions subject to one condition on the ratio of pressure gradient and skin friction which is discussed later. In particular, the theory is not confined to equilibrium boundary layers. The value of a theoretical analysis having such generality is evident, since there is only a small probability that the boundary layer in a given problem of engineering interest is of the equilibrium type.

The flows analyzed in this paper are simple but representative examples of turbulence in the vicinity of solid boundaries. It is expected that the method used here, say for a two-dimensional turbulent boundary layer, can be extended to more complicated three-dimensional boundary layers.

It turns out that the expansions used in the paper have a non-uniformity at the point of zero skin friction which is quite different from the non-uniform behaviour of laminar boundary layers. When $(dP_\infty/dx)/(RU_*^3)$ ceases to be small, errors become significant. (The notation is explained in §4.) The expansions are, in particular, not applicable to layers of zero skin friction.

The main approach of this paper is to extract as much information as possible from the underdetermined system of equations of mean motion without resorting to any closure hypothesis. This approach is somewhat similar to that of Millikan's well-known argument which established logical relationships among empirical correlations without invoking a hypothetical model of turbulence. The value of such a restrained approach lies in the reliability of the information obtained since the danger of oversimplifying the phenomena or of introducing extraneous features through a hypothetical model is absent.

2. Fully developed turbulent channel flow

Preliminaries

Consider a fully developed turbulent flow between two parallel plane smooth stationary walls of infinite extent. The fluid is assumed to be a Newtonian fluid of constant density and viscosity in this and subsequent sections. With the x axis in the downstream direction in the plane of symmetry, and the y axis normal to the plane, the relevant equations of mean motion can be written as

$$P_{,x} - (1/R)\,U_{,yy} - \tau_{xy,y} = 0, \tag{2.1a}$$

$$P_{,y} - \tau_{yy,y} = 0. \tag{2.1b}$$

$P(x, y; R)$ and $U(y; R)$ denote mean pressure and velocity non-dimensionalized by the use of h, the half-depth of the channel, as a reference length and U_0, the mean velocity in the plane of symmetry, as a reference velocity. $\tau(y; R)$ with appropriate suffixes denotes a component of non-dimensionalized Reynolds stress. R is the Reynolds number $U_0 h/\nu$. A comma followed by a suffix x or y denotes partial differentiation with respect to x or y. The boundary conditions on velocity components and their implications on Reynolds stresses are

$$y = \pm 1: U = u' = v' = \tau_{xy} = \tau_{xy,y} = \tau_{xy,yy} = 0. \tag{2.2}$$

Here u' and v' denote fluctuating components of velocity. The choice of the reference velocity requires that

$$U(0;\, R) = 1. \tag{2.3}$$

It is readily seen from (2.1) that $P_{,x}$ does not vary with y and is given by

$$P_{,x} = -(1/R)\,U_{,y}|_{y=-1} = -U_*^2, \tag{2.4}$$

where $U_*(R)$ is the non-dimensionalized friction velicity.

We now review some of the elementary notions of the method of matched asymptotic expansions in the context of expansions of mean velocity. Further reference can be made to the works of Cole (1968) and Van Dyke (1964). Mean velocity is assumed to have an asymptotic expansion of the form

$$U(y; R) = \sum_1^m E_n(R)\, U_n(y) + o(E_m); \tag{2.5a}$$

the E_n are functions of the Reynolds number and are called gauge functions. They are so arranged that $E_{n+1}/E_n \to 0$ as $R \to \infty$. The coefficients U_n depend on the co-ordinate. The partial sum of m terms is called the m term expansion. The difference between it and the mean velocity is of a higher order than E_m by the definition of asymptotic expansion. Substitution of expansions such as (2.5a) into the equations of a given problem leads to equations for terms of various orders. When one tries to satisfy the boundary conditions of the problem, no difficulty is encountered in one class of problems which are called regular perturbation problems. More frequently, one is not able to satisfy the boundary conditions as the expansions such as (2.5a) are not uniformly valid and such problems are called singular perturbation problems. When the difficulty is associated with a non-uniform limiting behaviour in thin layers, the method of matched asymptotic expansions is very appropriate. It consists of (a) the use of expansions such as (2.5) outside the layers, (b) the use of expansions based on stretched coordinates in the layers, and (c) a systematic matching procedure which essentially requires that there is an overlap region where both the expansions are valid.

We also assume that the Reynolds stress τ_{xy} admits an asymptotic expansion

$$\tau_{xy}(y;\, R) = \sum_1^m \Gamma_n(R)\, T_{xyn}(y) + o(\Gamma_m), \tag{2.5b}$$

where Γ_n and T_{xyn} are the nth gauge function and coefficient.

We will first show that the problem is a singular perturbation problem. The condition (2.3) requires that E_1 be of the order of unity and we take E_1 to be one without any loss of generality. The hypothesis that the viscous stress terms in (2.1a) are of a higher order than the Reynolds stress terms requires that Γ_1 be of the order of U_*^2. By taking Γ_1 to be equal to U_*^2, the equation of the lowest-order terms becomes

$$T_{xy1,\,y} + 1 = 0. \tag{2.6}$$

It is readily seen that the solution

$$T_{xy1} = B_1 - y, \tag{2.7}$$

where B_1 is a constant, cannot satisfy the boundary conditions (2.2) as the hypothesis and the expansions are not valid near the channel walls.

To supplement the outer expansions (2.5), the following inner expansions are assumed near the lower wall

$$U(y;\, R) = \sum_1^m \epsilon_n(R)\, u_n(\eta) + o(\epsilon_m), \tag{2.8a}$$

$$\tau_{xy}(y;\, R) = \sum_1^m \gamma_n(R)\, \tau_{xyn}(\eta) + o(\gamma_m), \tag{2.8b}$$

where η is the inner variable $(y+1)/\delta$, δ being a function of R. ϵ_n and γ_n are gauge functions and u_n and τ_{xyn} are coefficients. Lower case letters are used for the inner expansion and upper case letters are used for the outer expansion. A similar expansion for the upper wall layer would be required for the complete treatment and it can be readily written down by analogy.

It is now postulated that the lowest-order viscous stress terms are of the order of the Reynolds stress terms in the inner layer, i.e. $\epsilon_1/\delta R$ is of the order of γ_1. Without any loss of generality, let γ_1 be given by

$$\gamma_1 = \epsilon_1/\delta R. \tag{2.9}$$

Whether the lowest-order terms of (2.1) contain U_*^2 depends on the behaviour of $U_*^2\delta^2 R/\epsilon_1$. Instead of making a hypothesis about this point, we consider the mean vorticity transport equation obtained from (2.1). The lowest-order terms then yield

$$u_{1,\eta\eta\eta} + \tau_{xy1,\eta\eta} = 0 \tag{2.10}$$

and, on integration,

$$u_1 = a_1\eta + a_2\eta^2 - \int_0^\eta \tau_{xy1}(\eta')\,d\eta', \tag{2.11}$$

where a_1 and a_2 are constants and one constant of integration vanishes because of the boundary condition (2.2).

Matching of expansions

Inner and outer expansions are matched so that both of them are valid in an overlap region. The matching condition suggested by Van Dyke is used here and it can be symbolically stated as

$$\mathcal{O}_m \mathcal{I}_n(f) = \mathcal{I}_n \mathcal{O}_m(f) \quad (m, n = 1, 2, 3, \ldots), \tag{2.12}$$

where $\mathcal{I}_m(f)$ and $\mathcal{O}_m(f)$ represent the m term inner and outer expansions of $f(y; R)$. The left side of the above condition can be obtained by first writing the n term inner expansion of f in terms of the outer variable y, and then taking the m term outer expansion. The right side of the relation is similarly obtained by first taking the m term outer expansion of f written in terms of the inner variable η and then taking the n term inner expansion.

In perturbation problems, one often assumes a sequence of gauge functions and employs (2.12) to determine constants of integration in inner and outer expansions. In this particular problem, conditions on the gauge functions and restrictions on the functional forms of mean velocity and Reynolds stress are obtained by repeated application of (2.12) for various values of m and n.

$\mathcal{I}_1\mathcal{O}_1(\tau_{xy})$ is seen from (2.7) to be $U_*^2(B_1+1)$. (Refer to Van Dyke (1964) for details.) The condition (2.12) would then require that τ_{xy1} approach a constant as $\eta \to \infty$. Let

$$\tau_{xy1} \sim b_1 + b_2/\eta \quad \text{as} \quad \eta \to \infty, \tag{2.13}$$

where b_1 and b_2 are constants. For matching, γ_1 is of the order of U_*^2. Without loss of generality, let γ_1 be equal to U_*^2. Then

$$\epsilon_1/\delta R = U_*^2, \tag{2.14a}$$

$$b_1 = B_1 + 1. \tag{2.14b}$$

Now directing attention to U, we find that $\mathcal{I}_1\mathcal{O}_1(U)$ is $U_1(-1)$, the value of U_1 at the lower wall. It is seen from (2.11) and (2.13), that

$$\mathcal{I}_1(U) \sim \epsilon_1[\delta^{-2}a_2(1+y)^2 + \delta^{-1}(a_1-b_1)(1+y) - b_2\ln\{(1+y)/\delta\} - b_3], \tag{2.15}$$

where b_3 is a constant. $\mathcal{O}_1\mathcal{I}_1(U)$ can be independent of y as required for matching only if

$$a_2 = 0, \quad a_1 = b_1. \tag{2.16}$$

Then

$$\mathcal{O}_1\mathcal{I}_1(U) \sim (\epsilon_1 \ln\delta)\, b_2. \tag{2.17}$$

Matching condition (2.12) for $m = n = 1$ applied to mean velocity requires that $\epsilon_1 \ln\delta$ is of the order of unity. To obtain results in a form similar to the conventional form, let

$$\epsilon_1 \ln\delta = -k + kc\epsilon_1 + o(\epsilon_1), \tag{2.18}$$

where k and c are constants. Then matching requires that

$$b_2 = -U_1(-1)/k. \tag{2.19}$$

Now $\mathcal{O}_2\mathcal{I}_1(U)$ is seen from (2.15) to (2.19) to be

$$\mathcal{O}_2\mathcal{I}_1(U) \sim -\epsilon_1[U_1(-1)/k]\ln\delta + \epsilon_1[\{U_1(-1)/k\}\ln(1+y) - b_3]. \tag{2.20}$$

On the other hand, $\mathcal{O}_2(U)$ is

$$\mathcal{O}_2(U) = U_1(y) + E_2(R)\,U_2(y) \tag{2.21}$$

or, after taking a suitable expansion of U_2,

$$\mathcal{O}_2(U) \sim U_1(-1) + E_2[A_1\ln(1+y) + A_2 + A_3(1+y)^{-1}] \quad \text{as} \quad 1+y \to 0, \tag{2.22}$$

where A_1, A_2 and A_3 are constants. The matching condition (2.12) for $m = 2$, $n = 1$ requires that E_2 is of the order of ϵ_1. Let E_2 and ϵ_1 be equal without any loss of generality. Then matching requires that

$$A_1 = U_1(-1)/k, \quad A_2 = -cU_1(-1) - b_3. \tag{2.23}$$

One more relation is required between the two remaining gauge functions δ and ϵ_1. So the complete equation of kinetic energy of fluctuating motion is now considered. The terms describing the transfer from the mean motion, and the diffusion associated with fluctuations are $-\overline{u'v'}U_{,y}$ and $\frac{1}{2}[\overline{v'(u'^2+v'^2+w'^2)}]_{,y}$. The inner layer is by hypothesis characterized by a significant transfer of energy from the mean flow and an equally significant diffusive transfer through the fluctuating motion. If the fluctuating velocity components are of the order of $\gamma_1^{\frac{1}{2}}$, the lowest-order terms in the inner expansions of the above terms are of the orders of $\gamma_1\epsilon_1/\delta$ and $\gamma_1^{\frac{3}{2}}/\delta$. Hence, γ_1 is of the order of ϵ_1^2. Without any loss of generality, let γ_1 be equal to ϵ_1^2. Then it is seen from (2.9) and (2.14), that

$$\delta = 1/\epsilon_1 R = 1/U_* R. \tag{2.24}$$

The indirect dissipation rate in the equation of kinetic energy has terms like $R^{-1}\overline{u'_{,y}u'_{,y}}$ and hence it is of the order of $\gamma_1/\delta^2 R$ if the length scale of the fluctuating motion in the y direction is taken as δ. The alternative hypothesis that the indirect dissipation rate is of the order of the transfer from the mean motion leads to the conclusion that ϵ_1 is of the order of $1/\delta R$. It is seen from (2.24) that this hypothesis serves the purpose equally well.

The values of the constants a_1 and B_1 turn out to be unity and zero, as can be seen from (2.4), (2.11), and (2.24), and from (2.14b) and (2.16).

The viscous term in (2.1a) is of the order of $1/R$ in the outer expansion if $U_{1,yy}$ is not zero. But the viscous term was assumed to be of a higher order than U_*^2, and (2.18) and (2.24) imply that U_*^2 decreases considerably slower than $1/R$. Hence $U_{1,yy}$ must be zero. The outer flow is then symmetrical only if U_1 is independent of y. Hence $U_1(y)$ is equal to one.

Thus the outer expansions of mean velocity and Reynolds stress are given by

$$U = 1 + U_* U_2(y) + o(U_*), \tag{2.25a}$$

$$U \sim 1 + U_*[(1/k)\ln(1+y) + A_2] \quad \text{as} \quad y \to -1, \tag{2.25b}$$

$$\tau_{xy} = -U_*^2 y + o(U_*^2), \tag{2.25c}$$

$$\tau_{xy} \sim U_*^2 \quad \text{as} \quad y \to -1. \tag{2.25d}$$

The inner expansions are

$$U = U_*\left[\eta - \int_0^\eta \tau_{xy1}(\eta')\,d\eta'\right] + o(U_*), \tag{2.26a}$$

$$U \sim U_*[(1/k)\ln\eta + A_2 + c] \quad \text{as} \quad \eta \to \infty, \tag{2.26b}$$

$$\tau_{xy} = U_*^2\tau_{xy1}(\eta) + o(U_*^2), \tag{2.26c}$$

$$\tau_{xy} \sim U_*^2[1 - 1/k\eta] \quad \text{as} \quad \eta \to \infty. \tag{2.26d}$$

The relation (2.18) can now be written in a more familiar form by using (2.24),

$$1/U_* = (1/k)\ln(U_*R) + c + o(1). \tag{2.27}$$

The above solutions contain the undetermined functions U_2 and τ_{xy1} and the undetermined constants k, A_2 and c as the system being analyzed is underdetermined.

Relations (2.25*a*) and (2.26*a*) are the well-known velocity defect law and the law of the wall, while (2.25*b*) and (2.26*b*) are the logarithmic laws. Relation (2.27) is the well-known law for skin friction expressed in terms of the mean velocity at the plane of symmetry.

There are many ways of constructing uniformly valid expansions from the inner and the outer expansions. One such set of uniformly valid expansions is the following:

$$U = U_*\left[\eta - \int_0^\eta \tau_{xy1}(\eta')\,d\eta' + U_2(y) - (1/k)\ln(1+y) - A_2\right] + o(U_*), \tag{2.28a}$$

$$\tau_{xy} = U_*^2[\tau_{xy1}(\eta) - (1+y)] + o(U_*^2). \tag{2.28b}$$

Relation (2.28*a*) can be seen to be one form of the law of the wake.

It follows from (2.11) and (2.16) that

$$u_{1,\eta} + \tau_{xy1} = 1. \tag{2.29}$$

The above relation states that the total shear stress in the inner layer is constant to the lowest order. The reason for the constancy of total shear stress in the presence of pressure gradient is that the pressure gradient, being of the order of U_*^2, is of a higher order than the viscous stress terms in (2.1*a*), which are of the order of U_*^3R. The asymptotic conclusion (2.29) is a counterpart of the well-known constant total shear stress hypothesis.

Since the cross-sectional average of mean velocity is sometimes used in the skin friction law, it can be estimated by selecting η_1 in the overlap region and averaging (2.28*a*) as follows:

$$U_{\text{av}} = U_*\delta\int_0^{\eta_1}\left[\eta - \int_0^\eta \tau_{xy1}(\eta')\,d\eta'\right]d\eta + \int_{-1+\delta\eta_1}^0 [1 + U_*U_2(y)]\,dy + o(U_*)$$

or

$$U_{\text{av}} = 1 + c_1U_* + o(U_*), \tag{2.30}$$

where c_1 is a constant. The skin friction law (2.27) can now be written as

$$\sqrt{(2/C_f)} = (1/k)\ln\{R_{\text{av}}\sqrt{(\tfrac{1}{2}C_f)}\} + c + c_1 + o(1), \tag{2.31}$$

where R_{av} is the Reynolds number based on average velocity and the half-depth and C_f is the skin friction coefficient $2(U_*^2/U_{\text{av}}^2)$.

3. Fully developed turbulent pipe flow

The problem of a fully developed turbulent pipe flow is substantially similar to the channel flow problem. A few details are given below to show some of the minor differences.

With x, y, z as axial, radial and tangential co-ordinates, the fully developed parallel mean flow is governed by

$$P_{,x} - (1/R)\,D(U_{,y}) - D\tau_{xy} = 0, \tag{3.1a}$$

$$P_{,y} - D(\tau_{yy}) + \tau_{zz}/y = 0, \tag{3.1b}$$

where

$$D(f) = (yf)_{,y}/y. \tag{3.1c}$$

The notation used here is chosen to exploit similarities with the previous problem. The reference length and velocity used for non-dimensionalization are the radius of the pipe and the mean velocity at the axis. Equations (2.2) and (2.3) hold with the modification that the boundary condition (2.2) holds only at $y = +1$. The vorticity transport equation is simply

$$(1/R)\,[D(U_{,y})]_{,y} + (D\tau_{xy})_{,y} = 0. \tag{3.2}$$

The outer expansions (2.5) with the same hypothesis about the outer layer and also the choice of the gauge function E_1 gives the lowest-order equation for (3.1a) as

$$2 + DT_{xy1} = 0 \tag{3.3}$$

and, on integration,

$$T_{xy1} = -y + B_1/y. \tag{3.4}$$

The constant B_1 is taken to be zero for finite Reynolds stress at the axis.

The inner variable η is given by

$$\eta = (1-y)/\delta, \quad \delta = \delta(R), \tag{3.5}$$

which corresponds to the expansion for the upper wall in the channel flow problem. The inner expansions (2.8) with the same hypothesis about the inner layer and the choice of the gauge function (2.9) lead to the lowest-order equation (2.10). The differential operator D given by (3.1c) does not introduce any modifications in the lowest-order terms.

The remaining arguments and conclusions are the same as in the previous section except for two changes. The sign difference between the definition of the inner variable in the previous problem and (3.5) leads to obvious modifications in signs, and the integrands in (2.30) are multiplied by $2y$.

4. Two-dimensional turbulent boundary layers

We now consider the turbulent boundary layer past a flat smooth surface with a given flow condition at one section (mean velocity and Reynolds stress), and given mean velocity or equivalent mean pressure outside the boundary layer. The mean quantities are assumed to be independent of the spanwise co-ordinate z. The complete equations of mean motion are

$$U_{,x} + V_{,y} = 0, \quad U = \Psi_{,y}, \quad V = -\Psi_{,x}, \tag{4.1a}$$

$$UU_{,x} + VU_{,y} = -P_{,x} + (1/R)\,[U_{,xx} + U_{,yy}] + \tau_{xx,x} + \tau_{xy,y}, \tag{4.1b}$$

$$UV_{,x} + VV_{,y} = -P_{,y} + (1/R)\,[V_{,xx} + V_{,yy}] + \tau_{xy,x} + \tau_{yy,y}, \tag{4.1c}$$

where a reference length L and a reference velocity U_0 are used for non-dimensionalization. R is U_0L/ν. Other conditions, which anticipate that the equations for the leading terms are parabolic, are

$$y = 0, \quad x \geqslant x_0: U = V = u' = v' = w' = 0, \tag{4.2a}$$

$$x = x_0, \quad y > 0: U = U(y). \tag{4.2b}$$

Inviscid limit:

$$x \geqslant x_0, y \to 0: U \to U_\infty(x), \quad V \to 0, \quad P \to P_\infty(x). \tag{4.2c}$$

The mean vorticity transport equation is given by

$$U(\nabla^2\Psi)_{,x} + V(\nabla^2\Psi)_{,y} = (1/R)\nabla^4\Psi + \tau_{xx,xy} + \tau_{xy,yy} - \tau_{xy,xx} - \tau_{yy,xy}. \tag{4.3}$$

Expansion for the outer layer

The existence of the inner and outer layers is stipulated as in the channel and the pipe flow and the outer expansion (not for the region outside the boundary layer but for the outer layer) is taken in the form

$$Y = y/\Delta, \quad \Delta = \Delta(R), \tag{4.4a}$$

$$\Psi(x, y; R) = \Delta E_1(R)\Psi_1(x, Y) + \Delta E_2(R)\Psi_2(x, Y) + o(\Delta E_2), \tag{4.4b}$$

$$U(x, y; R) = E_1\Psi_{1,Y} + E_2\Psi_{2,Y} + o(E_2), \tag{4.4c}$$

$$\tau(x, y; R) = \Gamma_1(R)\,T_1(x, Y) + o(\Gamma_1). \tag{4.4d}$$

Here E_n, Γ_n are gauge functions and Ψ_n, T_n are coefficients.

An additional assumption required for turbulent boundary layers is that all the components of Reynolds stress are of the same order. Were it not so, the turbulence would become extremely anisotropic in the high Reynolds number limit, and certainly that is neither plausible nor in conformity with the experimental data.

The expansions of the terms of the mean vorticity equation are recorded for convenience

$$U(\nabla^2\Psi)_{,x} + V(\nabla^2\Psi)_{,y} = (E_1^2/\Delta)[\Psi_{1,Y}\Psi_{1,xYY} - \Psi_{1,x}\Psi_{1,YYY}] + (E_1E_2/\Delta) \times [\Psi_{1,Y}\Psi_{2,xYY} + \Psi_{2,Y}\Psi_{1,xYY} - \Psi_{1,x}\Psi_{2,YYY} - \Psi_{2,x}\Psi_{1,YYY}] + o(E_1E_2/\Delta), \tag{4.5a}$$

$$(1/R)\nabla^4\Psi = (E_1/\Delta^3R)\Psi_{1,YYYY} + o(E_1/\Delta^3R), \tag{4.5b}$$

$$\tau_{xx,xy} + \tau_{xy,yy} - \tau_{xy,xx} - \tau_{yy,xy} = (\Gamma_1/\Delta^2)\,T_{xy1,YY} + o(\Gamma_1/\Delta^2). \tag{4.5c}$$

It is postulated that the dominant character of the outer layer is inertial and that the Reynolds terms contribute not to the lowest but to the second lowest-order terms and that the viscosity terms contribute to still higher-order terms. In other words,

$$\Gamma_1/\Delta E_1E_2 \to \text{non-zero constant}, \quad 1/\Delta^2E_2R \to 0 \quad \text{as} \quad R \to \infty. \tag{4.6}$$

Although this assumption is not identical to that for the channel flow, it still recognizes that the Reynolds stress terms are more dominant than the viscosity terms.

The equations for the lowest and the second lowest order are then

$$\Psi_{1,Y}\Psi_{1,xYY}-\Psi_{1,x}\Psi_{1,YYY}=0, \tag{4.7a}$$

$$\Psi_{1,Y}\Psi_{2,xYY}+\Psi_{2,Y}\Psi_{1,xYY}-\Psi_{1,x}\Psi_{2,YYY}-\Psi_{2,x}\Psi_{1,YYY}=T_{xy1,YY}, \tag{4.7b}$$

where Γ_1 has been taken to be equal to $\Delta E_1 E_2$ without any loss of generality. Integration of (4.7a) gives

$$\Psi_{1,Y}\Psi_{1,xY}-\Psi_{1,x}\Psi_{1,YY}=H_1(x). \tag{4.8}$$

At the outer edge ($Y\to\infty$), the inviscid behaviour is given by

$$U_\infty U_{\infty,x}=-P_{\infty,x} \tag{4.9}$$

and matching with the inviscid flow requires that

$$U(x,Y)\to U_\infty(x) \quad \text{as} \quad Y\to\infty. \tag{4.10}$$

Hence H_1 is equal to $-P_{\infty,x}$.

Now E_1 has to be of the order of unity to satisfy (4.10). Let E_1 be unity. A solution of (4.7a) subject to (4.10) is

$$\Psi_1=U_\infty Y, \quad \Psi\sim\Delta U_\infty Y, \tag{4.11a}$$

$$U\sim U_\infty(x), \quad V\sim-\Delta U_{\infty,x}Y. \tag{4.11b}$$

As Y approaches zero, V approaches zero but U does not. This suggests a non-uniformity.

Any function of x alone can be added to Ψ_1 given by (4.11) and Ψ_1 will continue to be a solution of (4.11). However, such eigensolutions that give rise to normal velocity V at the inner and outer edge of the layer are thought to be irrelevant to the problem. Substitution of (4.11) in (4.7b) yields

$$U_\infty\Psi_{2,xYY}-YU_{\infty,x}\Psi_{2,YYY}=T_{xy1,YY}. \tag{4.12}$$

Integration yields,

$$U_\infty\Psi_{2,xY}+U_{\infty\ x}(\Psi_{2,Y}-Y\Psi_{2,YY})=T_{xy1,Y}+H_2(x). \tag{4.13}$$

As $Y\to\infty$, $\Psi_{2,xY}$, $\Psi_{2,Y}$, $Y\Psi_{2,YY}$ and $T_{xy1,Y}$ are expected to approach zero. So H_2 may be taken to be zero. Further integration yields

$$U_\infty\Psi_{2,x}+U_{\infty,x}[2\Psi_2-Y\Psi_{2,Y}]=T_{xy1}+H_3(x). \tag{4.14}$$

Here H_3 may not be zero as Ψ_2 may not approach zero as Y approaches infinity.

Expansion for the inner layer

Now consider the inner expansion

$$\eta=y/\delta, \quad \delta=\delta(R), \tag{4.15a}$$

$$\Psi(x,y;\,R)=\delta\epsilon_1(R)\,\psi_1(x,\eta)+\delta\epsilon_2(R)\,\psi_2(x,\eta)+o(\delta\epsilon_2), \tag{4.15b}$$

$$\tau(x,y;\,R)=\gamma_1(R)\,\tau_1(x,\eta)+o(\gamma_1), \tag{4.15c}$$

$$U(x,y;\,R)=\epsilon_1\psi_{1,\eta}+\epsilon_2\psi_{2,\eta}+o(\epsilon_2). \tag{4.15d}$$

A lower case letter in this notation has a meaning similar to the corresponding upper case letter used in the outer expansion. Expressions similar to (4.5) can be obtained by replacing $\Delta, E_n, \Gamma_n, \Psi_n$ and T_n by δ, ϵ_n, γ_n, ψ_n and τ_n respectively.

It is now stipulated, as in the previous problems, that the Reynolds stress terms are of the order of the viscosity terms, and that the inertia terms are of a higher order. That is, $\gamma_1 \delta R/\epsilon_1 \to$ a non-zero constant, and $\epsilon_1 \delta^2 R \to 0$. Let γ_1 be equal to $\epsilon_1/\delta R$. Then the lowest-order terms in (4.3) give

$$\psi_{1,\eta\eta\eta\eta} + \tau_{xy1,\eta\eta} = 0. \tag{4.16}$$

Integration and the use of the boundary conditions give

$$\psi_1(x,\eta) = f_1(x)\,\eta^2 + f_2(x)\,\eta^3 - \int_0^{\eta}\int_0^{\eta'} \tau_{xy1}(\eta'')\,d\eta''\,d\eta', \tag{4.17}$$

where f_1 and f_2 are unspecified functions of x.

Matching of the inner and the outer solutions

Consider the matching condition (2.12) for $m = n = 1$ applied to Reynolds stress τ_{xy}. Let $T_{xy1}(x, 0)$ be different from zero. Then $\mathscr{I}_1\mathscr{O}_1(\tau_{xy})$ is $\Gamma_1 T_{xy1}(x, 0)$. Hence γ_1 has to be of the order of Γ_1 and $\tau_{xy1}(x,\eta)$ approaches a limiting value as $\eta \to \infty$. Let γ_1 be equal to Γ_1 and

$$\tau_{xy1}(x,\eta) \sim g_1(x) + g_2(x)/\eta \quad \text{as} \quad \eta \to \infty, \tag{4.18}$$

where g_1 and g_2 are unspecified functions of x. Then from (2.12)

$$T_{xy1}(x, 0) = g_1(x). \tag{4.19}$$

$\mathscr{I}_1(U)$ is given from (4.17) and (4.18) as

$$\mathscr{I}_1(U) \sim \epsilon_1[(\delta/\Delta)^{-2}\, 3f_2\, Y^2 + (\delta/\Delta)^{-1}(2f_1 - g_1)\, Y - g_2 \ln\{Y/(\delta/\Delta)\} - h], \quad \eta \to \infty. \tag{4.20}$$

where h is a function of x. The matching condition (2.12) applied to mean velocity with $m = n = 1$ gives

$$f_2 = 0, \quad 2f_1 = g_1 \tag{4.21}$$

and $\epsilon_1 \ln(\delta/\Delta)$ has to be of the order of unity. In order to obtain results in a convenient form, let

$$\epsilon_1 \ln(\Delta/\delta) = k(U_{\infty 0} - c\epsilon_1) + o(\epsilon_1). \tag{4.22}$$

Here $U_{\infty 0}$ is the value of U_∞ at $x = x_0$, and k and c are constants. Then the condition (2.12) yields

$$g_2 = -(U_\infty/U_{\infty 0})/k. \tag{4.23}$$

Now $\mathscr{O}_2\mathscr{I}_1(U)$ is given by

$$\mathscr{O}_2\mathscr{I}_1(U) \sim \epsilon_1(U_\infty/U_{\infty 0})(1/k)\ln(\Delta/\delta) + \epsilon_1[(U_\infty/U_{\infty 0}k)\ln Y - h]. \tag{4.24}$$

The matching condition (2.12) with $m = 2, n = 1$ requires that E_2 be of the order of ϵ_1 and

$$\Psi_{2,Y} \sim F_1(x)\ln Y + F_2(x) + F_3(x)\, Y \quad \text{as} \quad Y \to 0. \tag{4.25}$$

Here F_n are unspecified functions of x. Let E_2 be equal to ϵ_1. Then matching requires that

$$U_\infty + \epsilon_1 \ln(\delta/\Delta)\, F_1 = o(1) \tag{4.26}$$

or F_1 can be taken to be U_∞/k in view of (4.22), and

$$F_2+(U_\infty/U_{\infty 0})\,c = -h. \tag{4.27}$$

As before, the complete equation of the kinetic energy of the fluctuating motion is now considered for the inner layer. The transport due to convection associated with mean motion is

$$\tfrac{1}{2}[U(\overline{u'^2+v'^2+w'^2})_{,x}+V(\overline{u'^2+v'^2+w'^2})_{,y}]$$

and is of the order of $\epsilon_1\gamma_1$. The transport associated with the fluctuating motion is given by

$$\tfrac{1}{2}[-\{\overline{u'(u'^2+v'^2+w'^2)}\}_{,x}-\{\overline{v'(u'^2+v'^2+w'^2)}\}_{,y}]$$

and is of the order of $\gamma_1^{\frac{3}{2}}/\delta$ if all fluctuating velocities are assumed to be of the order of $\gamma_1^{\frac{1}{2}}$. The production of the kinetic energy from the mean motion is given by

$$-[\overline{u'^2}U_{,x}+\overline{u'v'}U_{,y}+\overline{u'v'}V_{,x}+\overline{v'^2}V_{,y}]$$

and is of the order of $\gamma_1\epsilon_1/\delta$. The transfer of the kinetic energy is again assumed to be as significant as its production in the inner layer. Since the convection associated with mean motion is definitely of a higher order, $\gamma_1\epsilon_1/\delta$ is assumed to be of the order of $\gamma_1^{\frac{3}{2}}/\delta$. Let γ_1 be equal to ϵ_1^2. Then

$$\epsilon_1 = \Delta = 1/\delta R. \tag{4.28}$$

The non-dimensionalized friction velocity $U_*(x)$ is given by

$$U_*^2(x) = (1/R)\,U_{,y}|_{y=0} \sim [2\epsilon_1/\delta R]f_1(x) \sim 2\epsilon_1^2 f_1(x). \tag{4.29}$$

Hence if $U_*(x_0)$ is not zero, we may take

$$\epsilon_1 = U_*(x_0) = U_{*0}, \tag{4.30}$$

so that $2f_1(x_0)$ is unity. Note further that from (4.1), (4.2) and (4.21)

$$P_{\infty,x} = (1/R)\,U_{,yy}|_{y=0} = (\epsilon_1/\delta^2 R)\,(6f_2)+o(\epsilon_1/\delta^2 R) = o(U_{*0}^3 R). \tag{4.31}$$

Since U_{*0} decreases extremely slowly, U_{*0}^3R approaches infinity and the condition (4.21) on f_2 due to matching is therefore not very stringent. This aspect of the theory is discussed in a later section.

The outer expansions can be written as

$$U = U_\infty(x)+U_{*0}\Psi_{2,Y}(x, Y)+o(U_{*0}), \tag{4.32a}$$

$$U \sim U_\infty(x)+U_{*0}[(U_\infty/U_{\infty 0})\,(1/k)\ln Y+F_2(x)] \quad \text{as} \quad Y\to 0, \tag{4.32b}$$

$$\tau_{xy} = U_{*0}^2 T_{xy1}(x, Y)+o(U_{*0}^2), \tag{4.32c}$$

$$\tau_{xy} \sim U_{*0}^2[2f_1(x)] \quad \text{as} \quad Y\to 0. \tag{4.32d}$$

The inner expansions are

$$U = U_{*0}\psi_{1,\eta}(x,\eta)+o(U_{*0}), \tag{4.33a}$$

$$U \sim U_{*0}[(U_\infty/U_{\infty 0})\,(1/k)\ln\eta+F_2(x)+(U_\infty/U_{\infty 0})\,c] \quad \text{as} \quad \eta\to\infty, \tag{4.33b}$$

$$\tau_{xy} = U_{*0}^2\tau_{xy1}(x,\eta)+o(U_{*0}^2), \tag{4.33c}$$

$$\tau_{xy} \sim U_{*0}^2[2f_1(x)-U_\infty/(U_{\infty 0}k\eta)] \quad \text{as} \quad \eta\to\infty. \tag{4.33d}$$

Relations (4.32a) and (4.33a) correspond to the velocity defect law and the law of the wall. Equations (4.32b) and (4.33b) correspond to the logarithmic law in the overlap region. Further discussion on these relations is given in a later section.

Ψ_2 is connected with T_{xy1} by relation (4.14) and ψ_1 is written in terms of τ_{xy1} in (4.17). Thus relations (4.32a, c) and (4.33a, c) contain two undetermined functions $\Psi_2(x, Y)$ and $\tau_{xy1}(x, \eta)$. The logarithmic laws also contain an unknown function of x and two constants. This feature is a consequence of the underdetermined nature of the system of equations. Note further that the skin friction coefficient $C_{f0} = 2U_{*0}^2/U_{\infty 0}^2$ is given from (4.22) by

$$\sqrt{(2/C_{f0})} = (1/k)\ln\{(U_{\infty 0}\Delta R)\sqrt{(C_f/2)}\}+c+o(1). \quad (4.34)$$

Note that $U_{\infty 0}\Delta R$ is the Reynolds number based on external velocity at x_0 and the boundary-layer thickness Δ. The relation with the usual skin friction law will also be discussed later.

Uniformly valid expansions can be written as

$$U = U_{*0}[\psi_{1,\eta}(x,\eta)+\Psi_{2,Y}(x,y)-(U_\infty/U_{\infty 0})(1/k)\ln Y - F_2]+o(U_{*0}), \quad (4.35a)$$

$$\tau_{xy} = U_{*0}^2[\tau_{xy1}(x,\eta)+T_{xy1}(x, Y)-2f_1(x)]+o(U_{*0}^2). \quad (4.35b)$$

Other results

Often it is more convenient to express mean velocity and shear stress in terms of local friction velocity U_* and local boundary-layer thickness $\Delta'(x; R)$. Relations (4.32) and (4.33) accommodate such a change provided a certain vicinity of the separation point is avoided. Note also that when $U_*(x; R)$ is used in the place of a gauge function, it hides the parameter perturbation character of the problem. Its advantage, however, lies in a rapid indication of any tendency towards a similar profiles. Equations (4.32a) and (4.33a) now appear as

$$U(x,y;\, R) = U_\infty(x) - U_*(x;\, R)\,F(x, Y')+o(U_{*0}), \quad (4.36a)$$

$$U(x,y;\, R) = U_*(x;R)f(x, Y')+o(U_{*0}^2). \quad (4.36b)$$

Here $F(x, Y')$ and $f(x, Y')$ are given by $(U_{*0}/U_*)(-\Psi_{2,Y})$ and $(U_{*0}/U_*)\psi_{1,\eta}$ $\Delta'(x; R)$ is conveniently defined by

$$\Delta'(x;\, R) = \int_0^\infty (U_\infty - U)/U_*\,dy = \delta^* U_\infty/U_*, \quad (4.36c)$$

where δ^* is the displacement thickness. The new outer variable is given by

$$Y' = y/\Delta'. \quad (4.36d)$$

As a result,

$$\int_0^\infty F(x, Y')\,dY' = 1. \quad (4.36e)$$

With the expansion (4.32c) similarly rewritten, (4.13) becomes after some simplification

$$(\Delta'/U_*)\,U_{\infty,x}[-F+Y'F_{,Y'}]-(U_\infty\Delta'/U_*)\,[F_{,x}-(Y'\Delta'_{,x}/\Delta')\,F_{,Y'}] - (U_\infty/U_*^2)\,\Delta' F U_{*,x} = T'_{xy1,Y'}. \quad (4.37)$$

With the notation

$$\omega = U_*/U_\infty \quad (4.38a)$$

and

$$\pi(x;\, R) = \delta^* P_{,x}/U_*^2 = -\Delta' U_{\infty,x}/U_*, \quad (4.38b)$$

the relation (4.37) can be written as

$$\pi[2F - Y'F_{,Y'}] + [\delta^*_{,x}/\omega^2 - (\delta^*/\omega^3)\,\omega_{,x}]\,Y'F_{,Y} - (\delta^*/\omega^2)\,F_{,x} - (\delta^*/\omega^3)\,\omega_{,x}F = T'_{xy1,Y'}, \tag{4.38c}$$

which corresponds to the usual equation (see Rotta 1962, for example (15.8)) to the lowest order. The conventional equations have terms of two orders together since a limiting argument is not used.

Two notions of similarity of the mean velocity defect profiles can now be considered. If $(U_\infty - U)/U_*$ depends only on $y/\Delta'(x)$, the profile shapes are exactly similar and, as Clauser (1956) has shown, such exact similarity is unlikely to occur. An examination of his arguments shows that the slight differences found by him are due to higher-order terms. The weaker notion of similarity would require F to depend only on y/Δ'. This notion requires similarity only to the order of U_* and is more suitable for turbulent boundary layers.

A necessary condition for weak similarity is obtained from (4.38c), namely, π, $\delta^*_{,x}/\omega^2$, and $(\delta^*/\omega^3)\,\omega_{,x}$ are independent of x. It is then readily seen that

$$\omega = A(x-x_1)^n, \quad \delta^* = B(x-x_1)^{2n+1}, \tag{4.39}$$

with some constants A, B, x_1 and n. Also T'_{xy1} has to be independent of x. Thus according to the weak notion of similarity, the expansions of mean velocity defect and Reynolds stress up to the orders of U_* and U_*^2 depend on only y/Δ'.

A comparison of the coefficient of the logarithmic term in the expansion for $Y \to 0$ of (4.36a) with that of (4.32b), yields a condition stronger than (4.39) namely that ω is independent of x. Consequently, the logarithmic laws (4.32b) and (4.33b) reduce to the familiar forms,

$$U \sim U_\infty(x) + U_*[(1/k)\ln Y' + A_1] \quad \text{as} \quad Y' \to 0, \tag{4.40a}$$

$$U \sim U_*(x)\,[(1/k)\ln \eta' + A_1 + c] \quad \text{as} \quad \eta' \to \infty, \tag{4.40b}$$

where A_1 is a constant. The skin friction law then becomes

$$\surd(2/C_f) = (1/k)\ln[(U_\infty \Delta' R)\,\surd(C_f/2)] + c + o(1). \tag{4.40c}$$

For the purpose of approximate calculations, (4.38c) can be integrated after multiplying by a weighting function. Consider the integral equation for this range $Y' = 0$, $Y' \to \infty$ when the weighting function is unity. From the behaviour of U in the overlap region,

$$3\pi - \Delta'_{,x}/\omega - (\Delta'/\omega)\,\omega_{,x} = -1. \tag{4.41}$$

This equation also corresponds to the usual equation (Rotta 1962, (14.6)) to the lowest order.

5. Discussion

The hypotheses

The physical content of the major hypotheses can be described as follows. The balance of mean vorticity in the outer layer is dominated by convection associated with the mean motion. Reynolds stress terms influence the next order and viscous diffusion enters only in still higher-order terms. This region has a dominant historical character.

Viscous diffusion is as significant as Reynolds stress terms in the mean vorticity balance of the inner layer. However, the convection of mean vorticity associated with mean motion is of a higher order. Also, the production of kinetic energy of the fluctuating motion from the mean motion and its transfer associated with the fluctuating motion are equally important. Alternately, the production terms can be assumed to be of the order of the indirect dissipation terms. It is now widely recognized that the region close to the wall has these important properties.

The hypothesis that the Reynolds stress components or the fluctuating velocity components are of the same order in the inner or the outer layer, is very plausible. Otherwise very striking anisotropy would be observed. The root-mean-square values of the fluctuating velocity components and other data indicate that there is only a limited amount of anisotropy in the inner layer and considerably less in the outer layer.

For the pipe and the channel flows, convection of vorticity associated with mean motion happens to be absent. The above statements can then be readily modified.

These hypotheses are rather mild and describe properties which are known to be broadly supported by experimental data.

The method

The method of matched asymptotic expansions has been extensively used in diverse laminar flow problems and also in other problems of applied mathematics (Cole 1968; Van Dyke 1964). It provides a scheme for constructing uniformly valid solutions. It is based on the essential features of the classical boundary-layer arguments. The common order-of-magnitude arguments provide an alternative and conceptually simpler way of obtaining lowest-order equations, particularly in simple problems. However, numerous controversies have arisen from the use of such arguments. The method of matched asymptotic expansions seems to be considerably more reliable and mathematically more satisfactory.

The results

The form of the various asymptotic relations corresponds broadly to the various empirical laws. The most important difference is the asymptotic nature. The empirical laws have not been interpreted as asymptotic laws although some small systematic departures have been noted. For example, Rotta (1962, p. 101) finds a higher-order effect in the velocity defect law. Similarly, the local skin friction *vs.* local Reynolds number curve based on the data of Schultz-Grunow (1956) and Smith & Walker (1959) indicate a slight departure from a strictly logarithmic behaviour (Rotta 1962, p. 104).

The most specific result is the logarithmic laws containing two undetermined constants (4.34) and (4.40). Millikan's (1938) argument has shown that they are a consequence of the overlap of the two layers. The preceding arguments have shown that the physical nature of the layers can be postulated in broad terms and that the asymptotic form of velocity defect law and the law of the wall can be obtained in contrast to the exact laws assumed in Millikan's argument. Also, the results are not confined to similar profiles.

The role of pressure gradient

The total shear stress is constant to the order of U_*^2 in the inner layer as can be seen from (2.29), (4.17) and (4.21). This asymptotic conclusion is a counterpart of the well-known hypothesis of constant total shear stress. To see how pressure gradient gives rise to higher-order terms, consider the co-ordinate expansion of $U(x,y;\,R)$ near the wall.

$$\begin{aligned} U(x,y;\,R) &= RU_*^2 y + \tfrac{1}{2}P_{\infty,x}Ry^2 + o(y^2) \\ &= U_*[\eta' + \tfrac{1}{2}(1/RU_*^3)\,P_{\infty,x}\eta'^2] + o(y^2). \end{aligned} \tag{5.1a}$$

On the other hand, the inner expansion gives

$$U(x,y;R) = U_*\eta' + o(\eta'^2). \tag{5.1b}$$

Clearly, the effect of pressure gradient is governed by the term $P_{\infty,x}/(RU_*^3)$. For a given pressure distribution and a given location other than a point of zero friction, this term approaches zero as Reynolds number goes to infinity. In a separating boundary layer, however, this term does not approach zero uniformly, since at a given Reynolds number, however large it may be, there are points ahead of the separation point where the term $(P_{\infty,x}/U_*^3R)$ is large. As a result, the expansion used for the inner layer will be invalid near the separation point. In particular, the mean velocity profile in the overlap region may be expected to show marked deviation from the logarithmic law. Clearly, this non-uniformity is distinct from the non-uniform behaviour expected in laminar boundary layers (Goldstein 1948; Stewartson 1958; Kaplun 1967).

In channel or pipe flow, $P_{,x}(RU_*^3)$ is equal to $1/U_*R$ and hence the effect of pressure gradient is of a higher order in the inner layer.

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